Coplanar Tail-Chase Aerial Combat as a Differential Game

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A reduced-order version of the one-on-one aerial combat problem is studied as a pursuit-evasion differential game. The coplanar motion takes place at given speeds and given maximum available turn rates, and is described by three state equations which are equivalent to the range, bearing, and heading of one aircraft relative to the other. The purpose of the study is to determine those relative geometries from which either aircraft can be guaranteed a win, regardless of the maneuver strategies of the other. Termination is specified by the tail-chase geometry, at which time the roles of pursuer and evader are known. The roles are found in general, together with the associated optimal turn maneuvers, by solution of the differential game of kind. For the numerical parameters chosen, neither aircraft can win from the majority of possible initial conditions if the other turns optimally in certain critical geometries.

Introduction

THE "tail-chase" version of the aerial combat problem as studied in this paper requires that one aircraft be directly ahead of the other at the end of the combat encounter, with a "near-parallel" heading. The relative motion in this configuration is assumed to be slow enough to permit the forward-firing weapon system of the pursuing aircraft to be effective. Many refinements can obviously be made in this model of the weapon system. However, it is believed that this termination criterion is at least representative of realistic termination criteria for aircraft equipped with conventional guns or air-to-air missiles.

In a reduced-order mathematical model of the relative motion, the two aircraft are assumed to fly in the same plane at constant speeds and with bounded turn rates, so that the relative motion is described by only three variables: the relative position (range and bearing) and the relative heading. These are the dynamics used in the Game of Two Cars, first studied as a differential game by Isaacs.[1] In Isaacs' version of the problem, however, the roles of pursuer and evader were fixed a priori, and termination occurred at a specified "capture" range, independent of the bearing and heading angles. This model is therefore associated with relatively simple transversality conditions on the adjoint variables. The present version of the problem, in contrast, terminates when *either* aircraft is directly in front of the other, with the heading difference bounded. This means that the roles at termination depend on the relative bearing and heading, but not on the range. As in Isaacs' version, however, both pilots are assumed to have exact knowledge of the relative geometry and of the speed, turn-rate, and weapon characteristics of the other aircraft, insofar as these influence the optimal maneuvers of the pilots.

In this paper we consider for illustrative purposes a slower, more maneuverable aircraft engaged in combat with a faster, less maneuverable aircraft. The problem of principal interest is the determination of the roles of pursuer and evader as functions of the relative position and heading. That is, when should the faster aircraft pursue, and when should the slower aircraft pursue? In this context, the pursuing aircraft is maneuvering into a tail attack on his opponent, and conversely the evading aircraft maneuvers to frustrate a tail attack by the pursuing aircraft. In many situations, the role of each pilot can be intuitively predicted. Thus, for example, in the geometry of Fig. 1, when aircraft B lies well ahead of aircraft A, with a near-parallel heading, it could be assumed with reasonable certainty that A should pursue, and B should evade. Conversely, when aircraft B lies well to the rear in Fig. 1, the roles would be assumed to be reversed. However, at some intermediate point or set of points between these two extreme positions, intuition fails, and role specification is in doubt. Since certain initial conditions are then obvious "set ups" for one pilot or the other, it is clear that at least two regions must always exist in the relative geometry state space. These regions correspond to one of the following outcomes: 1) the faster aircraft (A) can win, regardless of the maneuvers used by the slower aircraft (B) and 2) the slower aircraft can win, regardless of the maneuvers used by the faster aircraft.

The present paper examines the tail-chase end condition in the aerial combat problem to establish the boundaries of the two capture regions. That is, in the language of differential game theory, a game of kind is solved. Now, the tail-chase end condition considered involves both the relative bearing and the relative heading, while the only control available to each of the two pilots is the turn rate. It can be seen by example that when the less maneuverable aircraft is sufficiently faster than the other, a third region must exist. In this region each aircraft can evade the other indefinitely, and it therefore is termed a "draw" region. In the similar problem analyzed in Ref. 2, both forward-firing weapons had finite

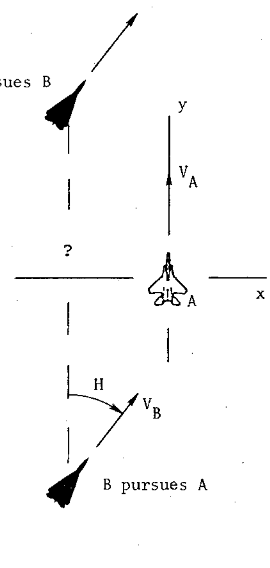

Fig. 1 The role determination problem.

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ranges and no terminal heading requirement was imposed. The draw region existed in this solution because the faster aircraft could remain out of range of the slower, which had the larger weapon range. In the present problem, if the faster aircraft were also more maneuverable, the draw region would presumably disappear, but this has not yet been verified quantitatively.

The main result desired from the solution to the problem is the geometric specification of the win regions of the two aircraft. Optimal (min-max) time strategies[3,4] of both pursuer and evader would be a natural next step in the problem when either A or B can win, but this extension is not undertaken here.

Method of Solution

The problem posed first requires specification of numerical values for aircraft speeds, maximum turn rates, and terminal relative heading angle requirements. State variable components[1] are the relative position (x,y), or the range and bearing of B with respect to A, and the relative heading (H). The optimal-turn maneuvers of both aircraft are to be found as functions of the relative geometry, using the methods described in Refs. 2 and 5. These techniques are briefly outlined as follows:

1) Several types of "near-miss" or simultaneous-kill end conditions are possible. For the near-miss end condition, one aircraft trajectory just contacts the kill region of the other, without penetrating it, and the roles of both pilots are known by construction at the time of this near miss. Similarly, the simultaneous-kill end condition corresponds to a collision with near-parallel headings. For nearby geometric states, one or the other aircraft wins *without* colliding with the other.

2) Families of optimal trajectories can be displayed as line segments which bound the kill regions and which are computed and shown at constant values of the relative heading. These "barrier"[1] line segments for the various potential maneuver combinations must be such as to separate relative positions leading to "capture" or to "escape". That is, on one side of the barrier, capture is guaranteed if the pursuer maneuvers optimally, and on the other side escape is guaranteed if the evader maneuvers optimally.

3) The state variables and the adjoint variables together imply optimal maneuvers of both aircraft, and relative trajectories can be found in retrograde time, even when switches occur in these backward-time paths.

Speed and turn-rate parameters in the present paper are such that the kill regions of both aircraft exist for only a finite range of headings. When the aircraft must move in the same plane, it is clear that the relative heading rate can be controlled by the aircraft with the higher maximum turn rate. But, the specified tail-chase end condition also requires that the relative angular *bearing* satisfy certain conditions, and the bearing rate depends on both speeds and the relative geometry (including range, bearing, and heading). Thus, even when the range is unconstrained, both pilots wish to control *two* of the state variables, but they have only their turn rates as input controls.

The three equations of relative motion, together with the three adjoint equations,[1,5] describe the maneuvers and the resulting motion of the slower, more maneuverable aircraft B relative to the faster aircraft A. That is, $V_A > V_B$, and $\omega_{A_{\max}} < \omega_{B_{\max}}$. The combat encounter (or "differential game") ends in favor of aircraft A when B is ahead of A with a nearly parallel heading; i.e., when the end conditions are

$$x_f = 0, \qquad y_f > 0, \qquad |H_f| \leq H_A \tag{1}$$

where H_A is a specified angle depending on the weapon system of aircraft A. On the other hand, B is the victor for the end conditions:

$$x_f = y_f \tan H_f, \qquad y_f < 0, \qquad |H_f| \leq H_B \tag{2}$$

The angular parameters H_A and H_B are meant to be descriptive of the tail-chase end condition geometry and are assumed to be of the order of 30 deg.

The optimal turn-rate controls of A and B are given in terms of the state $[x, y, H]$ and adjoint $[V_x, V_y, V_H]$ vectors, as follows:

$$\omega_A = \omega_{A_{\max}} \operatorname{sgn}[V_x y - V_y x + V_H], \quad \omega_B = \omega_{B_{\max}} \operatorname{sgn}[V_H] \tag{3}$$

The optimal controls thus depend on the present value of both the state vector and the adjoint vector. These can be determined simultaneously by integrating the equations of motion backwards in time from a range of geometric terminal conditions.

For the numerical parameters chosen, it is found that the assumption of infinite weapon range prevents singular arcs (straight dash trajectories) by either A or B from appearing in the solution. This would occur when either bracketed argument in Eq. (3) equals zero for a finite time. As noted earlier, the present mathematical model is a differential game of *angles*, while a singular arc for either aircraft would be associated with a trajectory which is at least locally *range* optimizing.[2]

Terminal Conditions

A total of 16 qualitatively different terminal configurations are applicable to this problem, of which six correspond to near-miss trajectories with A pursuing B, six to near-misses with B pursuing A, and four to collision of A and B. As defined earlier, the near-miss trajectories tangentially graze the kill zone of the pursuing aircraft. For all these configurations, the terminal adjoints can be determined as functions of the terminal state and the other parameters in the dynamic model.

The near-miss terminal geometry configurations are of three different types. The first type occurs at short range with both aircraft turning in the same direction, and with the evader always to the same side of the pursuer, as shown in Fig. 2a. The evader, B, is here just ahead of the pursuer, A, in position A_2B_2, but both before and after this time, B is to the right of A. The relative heading at this time is H_f, and this is *less* than the pursuer's maximum parametric value, H_A. The same type of trajectories can occur with the roles reversed; i.e., with B pursuing A.

The second type of grazing miss also involves turns in the same direction, but with the final relative heading *equal* to H_A. In this case the final range y_f is a parameter which must satisfy certain inequalities, but the appearance of the terminal trajectories is similar to those shown in Fig. 2a. The qualitative difference is that the bearing changes sign as B

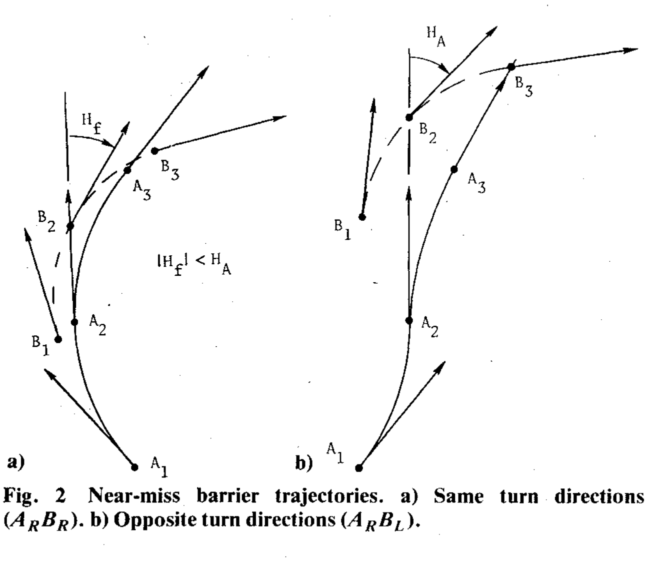


Fig. 2 Near-miss barrier trajectories. a) Same turn directions (A_RB_R). b) Opposite turn directions (A_RB_L).

Table 1 Terminal conditions for the near-miss configuration

Case	Pursuer	Final turn directions A	B	Final heading
1	A	Right	Right	$0 \leq H_f \leq H_A$
2		Right	Right	$H_f = H_A$
3		Right	Left	$H_f = H_A$
4		Left	Left	$-H_A \leq H_f \leq 0$
5		Left	Left	$H_f = -H_A$
6		Left	Right	$H_f = -H_A$
7	B	Right	Right	$0 \leq H_f \leq H_B$
8		Right	Right	$H_f = H_B$
9		Right	Left	$H_f = H_B$
10		Left	Left	$-H_B \leq H_f \leq 0$
11		Left	Left	$H_f = -H_B$
12		Left	Right	$H_f = -H_B$

passes from one side of A's velocity direction to the other. In this relatively complex case, the final adjoint vector is $[(\omega_A / V_A)\cos\beta,\ 0,\ \sin\beta]$, where $\tan\beta = (V_B \sin H_A - \omega_A y_f)\omega_A / [V_A(\omega_A - \omega_B)]$. Other terminal adjoints are given in Ref. 5.

The third type of near-miss grazing trajectory also ends at the maximum relative heading H_A, but is preceded by turns in the opposite directions. As shown in Fig. 2b, this means that the relative heading increases through the value H_A, and even if B can subsequently be kept ahead of A, the relative heading can be maintained by B at a value greater than H_A, because of B's greater maximum turn rate.

The turn directions and the pursuit-evasion roles of A and B both can be reversed, to provide a total of 12 terminal maneuvers and configurations for the near-miss end condition, as listed in Table 1.

The four collision geometric conditions correspond to terminal maneuvers in the same or opposite directions, and these configurations involve certain inequalities in the terminal heading which are specified in Fig. 3. These trajectories are interesting because, if either aircraft discontinues its turn slightly before the "collision," it will pass *ahead* of the other, with a near-parallel heading. Hence, by the definition of termination in this paper, the aircraft which seeks to avoid collision will be "killed" by the other. This aspect of the solution is of more than academic interest, because the collision course has a very real place in aerial combat, and is in fact one extreme hazard in aerial combat "hassles," where friendly aircraft engage in mock combat with each other. In this case, two aircraft which begin the hassle from a side-by-side position will usually enter a series of scissor maneuvers. The pilot who pulls harder will win such an encounter because he will force the other aircraft ahead of him at very short range. With two equally matched pilots and aircraft, there is a high risk of a collision unless one or the other pilot yields – and loses. Since fighter pilots are noted for their aggressive flying, there are many cases on record in which pilots and aircraft have been lost in this type of situation.

In the problem application considered by this paper between hostile pilots, the collision may therefore be the

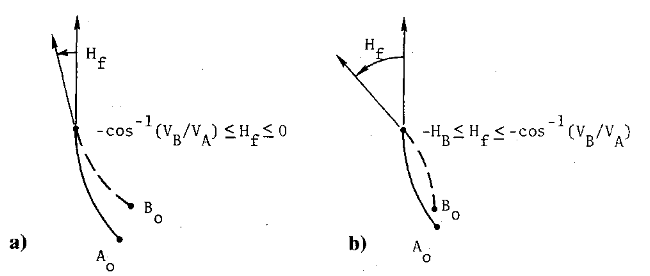


Fig. 3 Simultaneous kill barrier trajectories. a) Same turn directions ($A_R B_R$). b) Opposite turn directions ($A_L B_R$).

preferable outcome to both pilots, with each hoping the other will yield. As with other barrier trajectories, however, the collision barrier can be more practically interpreted as a locus across which the roles of pursuer and evader are reversed, and the representation of each aircraft as a *point* means that this locus in the three-dimensional state space is a two-dimensional *surface* separating distinct regions in this state space from one another.

Numerical Results

Retrograde solutions to the state and adjoint equations show where a grazing miss or a collision end condition must have originated. These solutions permit computation of the switch functions of both aircraft, according to Eq. (3), and these indicate when a retrograde switch occurs for either of the aircraft.

In the present case, for numerical purposes, the speeds and maximum turn rates of the two aircraft are taken as $V_A = 1400$ fps, $\omega_{A_{max}} = 10$ deg/sec, $V_B = 700$ fps, $\omega_{B_{max}} = 11$ deg/sec, while the terminal heading angular requirements are chosen as $H_A = 30$ deg and $H_B = 40$ deg. Thus, the slower aircraft B is slightly more maneuverable than A, while the normal accelerations in a hard turn are about 7.6 g and 4.2 g, respectively, for A and B. The minimum turn radii for A and B are 8020 ft and 3650 ft, respectively. These parameters do not apply to specific aircraft, but will guarantee that the "draw" condition occurs.

The barrier and role-specification results are most easily presented and interpreted by fixing the relative heading over a range of values between 0 deg and 180 deg. At each chosen value for the heading, barrier loci can then be calculated and presented using the concepts of the previous section. As shown in Fig. 4a for the parallel heading case, $H = 0$ deg, the loci subdivide the x-y space into an area where A pursues B, and an area where B pursues A. The implication of this is that when B is originally "outside" both A's capture region and B's capture region, neither can capture the other, and therefore each aircraft can evade the other indefinitely.

The following details of the barriers for the chosen numerical parameters are evident from a study of the graphical results shown in Fig. 4:

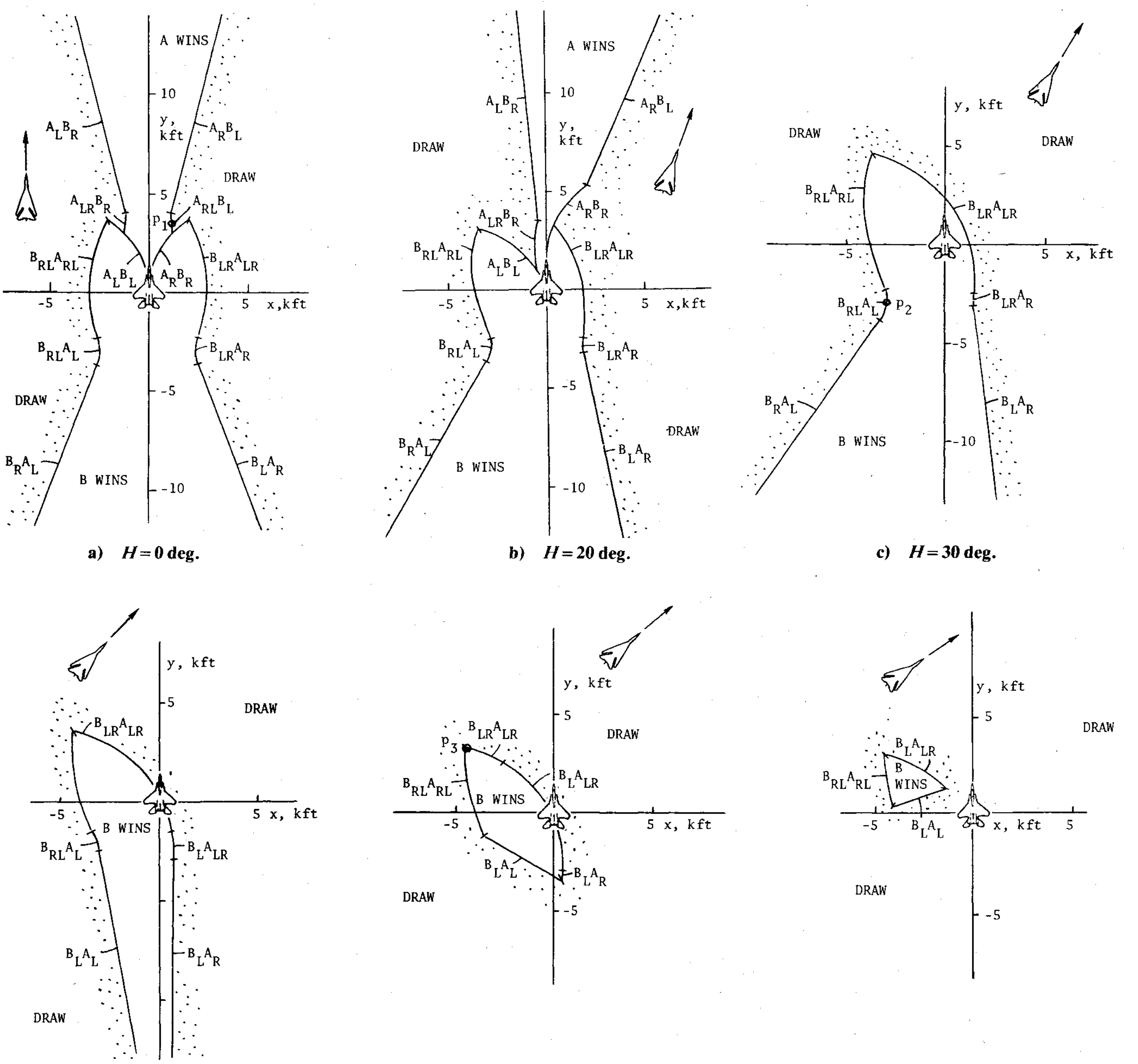
1) Many different maneuver combinations are required to completely define the barriers, several portions of which require two-part maneuvers. These are denoted by the double-subscript notation; e.g., $B_{RL}A_L$ means that B turns first right and then left in pursuing A, who turns left throughout the encounter which ends with A contacting B's kill region.

2) A can win from *some* relative positions only if the absolute initial heading is less than $H_A = 30$ deg, since otherwise B's higher turn rate can prevent its reduction to this value regardless of A's turns.

3) B can win from *some* relative positions only for absolute headings which are less than about 56 deg, because A's higher speed prevents B from controlling the bearing and the heading simultaneously when the heading exceeds this value.

4) Several "dispersal point" corners[1] occur in the barriers, where the slope is discontinuous, and where two *different* sets of maneuvers are optimal. The evader usually chooses the initial maneuver, forcing the pursuer to make a unique optimal response. Similarly, corners occur where B's kill region intersects A's kill region. At these points the roles are indeterminate, as for the collision barriers.

5) A's kill region is of infinite extent for headings less than H_A, due to the assumption of unbounded weapon range. Similarly, B's kill region is infinite for $H < 44$ deg, approximately. For small relative initial headings, the kill regions are bounded by straight lines at large ranges, corresponding to the one-stage near-miss maneuvers. In a refinement of the present mathematical model of the terminal conditions, these regions would be bounded at large range by curves which depend on the weapon ranges.



a) $H=0$ deg. b) $H=20$ deg. c) $H=30$ deg. d) $H=45$ deg. e) $H=50$ deg. f) $H=55$ deg.

Fig. 4 Barriers and capture regions in axes fixed to A. $V_A=1400$ fps; $\omega_{A_{max}}=10$ deg/sec; $H_A=30$ deg; $V_B=700$ fps; $\omega_{B_{max}}=11$ deg/sec; $H_B=40$ deg.

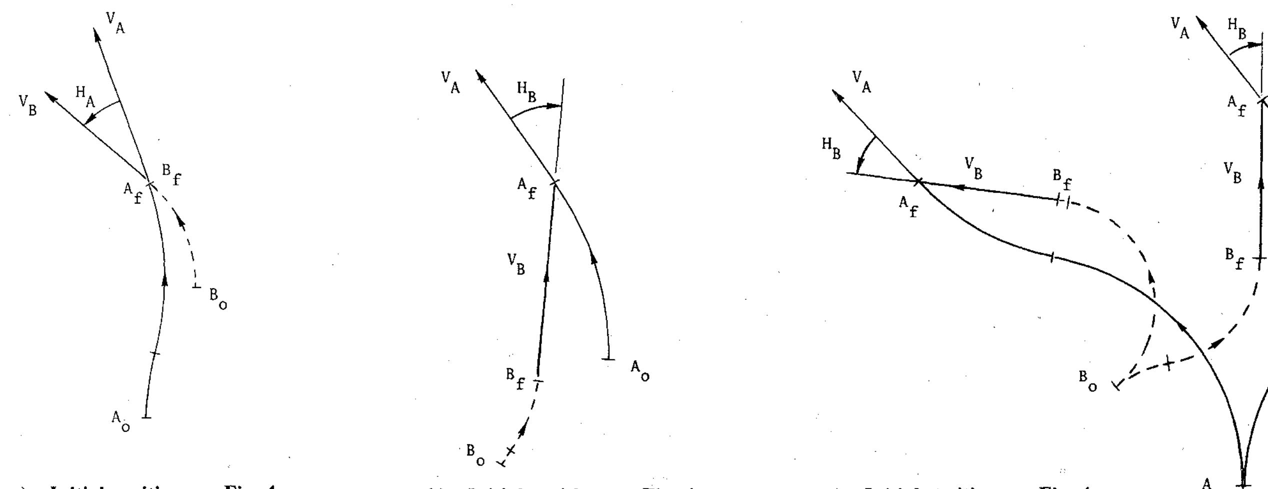
Four typical real-space barrier trajectories are illustrated in Fig. 5. The first of these corresponds to the initial location p_1 of Fig. 4a, and the second corresponds to the initial geometry p_2 of Fig. 4c. The last two trajectory pairs of Fig. 5 emanate from the dispersal point at p_3 in Fig. 4e. The two sets of paths are quite different, but both terminate with A ahead of B and with the final heading $H_f=\pm H_B$. These trajectories illustrate the variety and complexity of the barrier maneuvers, while the capture regions themselves can be used to answer the role-specification question posed in Fig. 1, which is the central question of interest in this paper.

In this "angular" pursuit-evasion game, the optimal barrier maneuvers are often obvious, and in many cases can be reduced to the guidance law: "Turn hard *toward* your opponent," regardless of the roles of the two aircraft. The exceptions to this simple rule can be noted for both A and B on the barrier segments of Fig. 4. These exceptions, along which the guidance law is "Turn hard *away*," occur only at short ranges, where optimal maneuvers are not obvious (see Fig. 1). These exceptions provide some of the important conclusions of the study.

Conclusions

The one-on-one aerial combat problem has been modeled as a differential game which ends with the pursuer headed directly at the evader, who is headed in nearly the same direction at this time. The analysis expresses the two optimal turn controls (ω_A and ω_B) as implicit functions of six vehicle/weapon parameters (V_A, V_B, $\omega_{A_{max}}$, $\omega_{B_{max}}$, H_A, and H_B) and as explicit functions of three state variables (x, y, H). Even in this idealized version of the problem, it is clear that all of these nine quantities should have some significance as to the roles and maneuvers of both aircraft. Furthermore, if the aircraft are at least qualitatively similar, either must be able to win from some relative conditions, as illustrated by the results obtained in the study.

The capture-regions in Fig. 4 show that the role-specification problem is particularly complex when each aircraft is inside the minimum-turn circle of the other. The strong dependence of the kill regions on the relative *heading* indicates that the magnitude and direction of the other's *velocity* must be estimated in applying these results.



a) **Initial position p_1, Fig. 4a.** b) **Initial position p_2, Fig. 4c.** c) **Initial position p_3, Fig. 4e.**

Fig. 5 Example barrier trajectories in real space.

If the weapon ranges are bounded, two more parameters must be specified, and additional complications enter the problem solution. As illustrated in Ref. 2 for weapon ranges which are large (but finite) compared to the turn radii, the capture regions are of finite extent at all relative headings. However, for practical values of the weapon ranges, the more complex and interesting barrier segments seem to be those which occur at short range; these barrier segments are independent of the weapon range parameters.

When both weapon ranges are finite, and when the faster aircraft is less maneuverable, *both* of the aircraft's actual kill regions will be finite. This means that the pilot who can apply the optimal *defensive* maneuvers can evade the other indefinitely, regardless of the other's offensive maneuvers. Eventually, a tactical error might permit an initially defensive pilot to take the offensive, should his kill region finally enclose the other aircraft. Other applications of the present method might deal with aircraft weapon performance trade-off studies, near-optimal maneuver specification for combat pilot training purposes, and Mach number-altitude variations of capture regions for a pair of opposing combat aircraft with their respective weapon systems.

The present dynamic model of the one-on-one combat problem is subject to two immediate criticisms. These are: 1) the constant velocities assumed for the aircraft, and 2) the coplanar motion assumed for the encounter. Both of these assumptions have been made in order to permit the third-order problem solution to be shown graphically, as in Fig. 4. However, the constant velocity assumption can be retroactively justified by the numerical results obtained. Specifically, all of the barrier trajectories are carried out at maximum turn rate, and usually for a time less than that required for a quarter-turn (Fig. 5). During such brief maneuvers, the aircraft speed would be kept nearly constant even if the pilot were not inclined to maintain the aircraft's energy. Furthermore, if it is found that the size and shape of the capture regions change slowly and continuously with the speeds, then practical use could be made of the results found over a range of constant speeds.

A variable-speed, noncoplanar model would require at least three more components of state to describe the relative position and velocity. Such a model could also incorporate more elaborate end conditions; e.g., minimum and maximum final ranges, finite final bearing intervals, zero final bearing rates, etc. The difficulty with such higher-order models is in the computation and presentation of the two capture regions. A particular problem would be the dispersal points, which must be found since the retrograde paths are not optimal beyond these points.

Although coplanar motion obviously cannot be justified on practical grounds, the actual terminal configuration occurs in a plane defined by the range vector (or the pursuer's velocity vector) and the evader's velocity vector. Optimal terminal accelerations also occur in this plane, and hence the paths just before termination are also coplanar. The optimal transition from arbitrary initial conditions to coplanar end conditions is a considerable generalization of the present problem which has not yet been studied. The solution to the coplanar problem, however, represents an easily understood and quantitative result which combines the important aircraft and weapon parameters and the coplanar position and heading coordinates to specify optimal maneuvers for both aircraft. This type of analysis thus develops an optimal *global* guidance law for an approximate dynamic model of the problem. This solution can be used to modify the approximate or intuitive guidance laws as applied to more exact dynamic equations. Both approaches must be exploited to develop a more complete understanding of problems of this complexity.

Acknowledgments

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